Entrance exam, April 2021, solutions

1. Simplify the expression $\left(\dfrac{a^{1/6}b^{-3}}{x^{-1}y}\right)^3 \cdot \left(\dfrac{x^{-2}b^{-1}}{a^{3/2}y^{1/3}}\right)$ and eliminate any negative exponent(s). Assume that all letters denote positive numbers.(5 points)

Solution: $a^{3(1/6)-(3/2)}b^{3(-3)+(-1)}x^{3\cdot 1-2}y^{3(-1)-(1/3)} = \dfrac{x}{ab^{10}y^{10/3}}$ (5 pts, no partial credit).

2. Factor the polynomials (a) x^4+2x^2+1; (b) x^4+4x^2+16. (4+8 points)

Solution: (a) $(x^2+1)^2$ (4 pts, no partial credit).
(b) $(x^2+4)^2-4x^2=(x^2+4+2x)(x^2+4-2x)$ (4+4 pts).

3. Solve the equation $\sqrt{x}-a\sqrt[3]{x}+b\sqrt[6]{x}-ab=0$ for the variable x. The constants a and b represent positive real numbers. (10 points)

Solution: We factor the left-hand side: $\sqrt[3]{x}(\sqrt[6]{x}-a)+b(\sqrt[6]{x}-a)=(\sqrt[3]{x}+b)(\sqrt[6]{x}-a)=0$ (4 pts), so one of the factors is 0 (1 pt). $\sqrt[6]{x}=a$ yields $x=a^6$ (2 pts). As $\sqrt[6]{x}$ makes sense only for $x\geq 0$, the other factor is positive, so it cannot be 0 (3 pts).

4. Jack, Kay, and Lynn deliver advertising flyers in a small town. If each person works alone, it takes Jack 6 h to deliver all the flyers, and it takes Lynn 3 h longer than it takes Kay. Working together, they can take all the flyers in 50% of the time it takes Kay working alone. How long does it take Kay to deliver all the flyers alone? (10 points)

Solution: Let k denote the number of hours Kay needs when working alone. Then Lynn needs $k+3$ hours. So, in one hour, Jack, Kay, and Lynn can deliver $1/6$, $1/k$, and $1/(k+3)$ parts of the flyers, resp. (2 pts). Working together, they need $k/2$ hours, so $\dfrac{k}{2}\left(\dfrac{1}{6}+\dfrac{1}{k}+\dfrac{1}{k+3}\right)=1$ (3 pts). After ordering, we get $k^2+3k-18=0$ (3 pts), the roots are 3 and -6 (1 pt). The negative root makes no sense, so Kay needs 3 hours when working alone (1 pt).

5. Solve the nonlinear inequality $\dfrac{3}{x-1}-\dfrac{4}{x}\geq 1$. Express the solution using interval notation. (10 points)

Solution: We transform the inequality into a form $0\leq f(x)$:

$$0\leq \frac{3}{x-1}-\frac{4}{x}-1=\frac{3x-4(x-1)-x(x-1)}{x(x-1)}=\frac{4-x^2}{x(x-1)}=\frac{(2-x)(2+x)}{x(x-1)}. \text{ (3 pts)}$$

The fraction makes sense for $x\neq 0,1$, and is non-negative if either the numerator is 0, or the numerator and the denominator both are positive or both are negative (1 pt). We get the information from the table (5 pts):

x	< -2	$= -2$	$\in (-2, 0)$	$\in (0, 1)$	$\in (1, 2)$	$= 2$	> 2
$4 - x^2$	$-$	0	$+$	$+$	$+$	0	$-$
$x(x - 1)$	$+$	$+$	$+$	$-$	$+$	$+$	$+$
$\frac{4-x^2}{x(x-1)}$	$-$	0	$+$	$-$	$+$	0	$-$

So, the solution is $[-2, 0) \cup (1, 2]$ (1 pt).

6a Find $f \circ g \circ h$. Here $f(x) = \sqrt{x}$, $g(x) = x/(x - 7)$, and $h(x) = \sqrt[4]{x}$. (4 points)

Solution:

$$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(\sqrt[4]{x}) = f\left(\frac{\sqrt[4]{x}}{\sqrt[4]{x} - 7}\right) = \sqrt{\frac{\sqrt[4]{x}}{\sqrt[4]{x} - 7}} \quad \text{(4 pts)}.$$

6b Find the inverse function of f. Here $f(x) = 2 + \sqrt{7 + x}$. Find the domain of f^{-1}. (7 points)

Solution: $y = 2 + \sqrt{7 + x} \iff (y - 2)^2 = 7 + x \iff x = (y - 2)^2 - 7$ (2 pts). So $f^{-1}(x) = (x-2)^2 - 7$ (2 pts). The domain of f^{-1} is the range of f. As the range of $\sqrt{7 + x}$ is the set of all non-negative real numbers, the range of f, i.e. the domain of f^{-1} is $[2, \infty)$ (3 pts).

6c Find a function f whose graph is a parabola with the vertex $(-1, 7)$ and that passes through the point $(-2, -2)$. (4 points)

Solution: Using the information about the vertex, we obtain that $f(x) = c(x + 1)^2 + 7$ for some constant c (2 pts). Plugging in $x = y = -2$, we obtain $-2 = c(-2 + 1)^2 + 7$, so $c = -9$, i.e. $f(x) = -9(x + 1)^2 + 7$ (2 pts).

7. Find the quotient and remainder using long division: $\dfrac{2x^5 - 7x^4 - 4}{4x^2 - 6x + 8}$. (10 points)

Solution: The quotient is $\frac{1}{2}x^3 - x^2 - \frac{5}{2}x - \frac{7}{4}$ and the remainder is $\frac{19}{2}x + 10$.

$$\begin{array}{cccccc|l}
2x^5 & -7x^4 & & & & -4 & : \ 4x^2 - 6x + 8 \ = \ \frac{1}{2}x^3 - x^2 - \frac{5}{2}x - \frac{7}{4} \\
2x^5 & -3x^4 & +4x^3 & & & & \\
\hline
 & -4x^4 & -4x^3 & & & -4 & \\
 & -4x^4 & +6x^3 & -8x^2 & & & \\
\hline
 & & -10x^3 & +8x^2 & & -4 & \\
 & & -10x^3 & +15x^2 & -20x & & \\
\hline
 & & & -7x^2 & +20x & -4 & \\
 & & & -7x^2 & +\frac{21}{2}x & -14 & \\
\hline
 & & & & \frac{19}{2}x & +10 &
\end{array}$$

8. Solve the equation $\log_8(x+2) + \log_8 3 = \log_8 8 + \log_8(2x-9)$ for x. (8 points)

Solution: The logarithms make sense only for $x+2>0$ and $2x-9>0$, i.e. $x>9/2$ (1 pt). Using the laws of the logarithm and the invertibility of the logarithmic function, we get the equivalent form $3(x+2) = 8(2x-9)$ (5 pts) if $x > 9/2$ (1 pt). The solution is $x = 6$ (1 pt).

9. Solve the equations for x:
(a) $\tan^{-1}(x) + \tan^{-1}(2x) = \pi/4$; (b) $\sin(5x) - \sin(3x) = \cos(4x)$. (10+10 points)

Solution: (a) Let $\tan^{-1}(x) = r$ and $\tan^{-1}(2x) = s$. This means $\tan r = x$, $\tan s = 2x$, and $r, s \in (-\pi/2, \pi/2)$ (2 pts). Rewriting the equation as $\pi/4 = r+s$, we take the tangent of both sides (this is not an equivalent step, as $-\pi/4 = r+s$ yields the same):

$$1 = \tan(r+s) = \frac{\tan r + \tan s}{1 - \tan r \cdot \tan s} = \frac{x+2x}{1 - x \cdot 2x} = \frac{3x}{1-2x^2}. \text{ (3 pts)}$$

Solving the arising quadratic equation $2x^2 + 3x - 1 = 0$, we get $x_1 = (-3+\sqrt{17})/4$ and $x_2 = (-3-\sqrt{17})/4$ (2 pts). However, the negative root does not satisfy the original equation, since then r and s are negative, so their sum cannot be positive (their sum is $-\pi/4$ in this case) (2 pts). Thus the only solution is $x = (-3+\sqrt{17})/4$ (1 pt).

(b) By the addition formulas, we have

$$\begin{aligned}\sin(5x) - \sin(3x) &= \sin(4x+x) - \sin(4x-x) = \\ &= \big(\sin(4x)\cos x + \cos(4x)\sin x\big) - \big(\sin(4x)\cos x - \cos(4x)\sin x\big) = 2\cos(4x)\sin x,\end{aligned}$$

so another form of our equation is $2\cos(4x)\sin x = \cos(4x)$ (3 pts). After ordering and factoring we get $\cos(4x)(2\sin x - 1) = 0$, i.e. (i) $\cos(4x) = 0$ or (ii) $\sin x = 1/2$ (3 pts). (i) holds if and only if $4x = \frac{\pi}{2} + k\pi$, i.e. $x = \frac{\pi}{8} + k\frac{\pi}{2}, k \in \mathbf{Z}$ (2 pts). From (ii) we obtain $x = \frac{\pi}{6} + 2k\pi$ and $x = \frac{5\pi}{6} + 2k\pi, k \in \mathbf{Z}$ (2 pts).